

Symmetry breaking in de Sitter: a stochastic effective theory approach

Gianrocco Lazzari* and Tomislav Prokopec*

*Institute for Theoretical Physics, Utrecht University,
Postbus 80.195, 3508 TD Utrecht, The Netherlands*

We consider phase transitions on (eternal) de Sitter in an $O(N)$ symmetric scalar field theory. Making use of Starobinsky's stochastic inflation we prove that deep infrared scalar modes cannot form a condensate – and hence they see an effective potential that allows no phase transition. We show that by proving convexity of the effective potential that governs deep infrared field fluctuations both at the origin as well as at arbitrary values of the field. Next, we present numerical plots of the scalar field probability distribution function (PDF) and the corresponding effective potential for several values of the coupling constant at the asymptotic future timelike infinity of de Sitter. For small field values the effective potential has an approximately quadratic form, corresponding to a positive mass term, such that the corresponding PDF is approximately Gaussian. However, the curvature of the effective potential shows qualitatively different (typically much softer) behavior on the coupling constant than that implied by the Starobinsky-Yokoyama procedure. For large field values, the effective potential as expected reduces to the tree level potential plus a positive correction that only weakly (logarithmically) depends on the background field. Finally, we calculate the backreaction of fluctuations on the background geometry and show that it is positive.

I. INTRODUCTION

The physics of de Sitter space is hard because there is no simple perturbative expansion parameter. In particular, massless scalars and gravitons exhibit so much particle production that their interactions cannot be perturbatively controlled. This breakdown of perturbative expansion is a serious obstacle to progress of our understanding of the physics of de Sitter space. The usual resummation techniques do not typically help. One such resummation is the self-consistent Hartree (mean field) approximation [1], which includes the one-loop resummation of daisy (and superdaisy) diagrams. When applied to an $O(N)$ symmetric real

* E-mail: g.lazzari@students.uu.nl; t.prokopec@uu.nl

scalar field on de Sitter space, this approximation scheme erroneously predicts that the $O(N)$ symmetry gets completely broken in the vacuum, and the (would-be) Goldstone bosons acquire a mass [1]. To get this result it is sufficient to assume the self-consistent Hartree and the de Sitter symmetry. A notable observation is that the mass of the Goldstones is strictly smaller than the mass of the condensate field. Recently, a more sophisticated resummation scheme based on a large N expansion in an $O(N)$ symmetric model has been utilized in Refs. [2, 3] to study deep infrared correlators on de Sitter space. Furthermore, based on the Euclidean approach to de Sitter, progress has been also made towards understanding mass generation in an interacting scalar field theory with a quartic self-interaction and one real scalar field (the $O(1)$ model) [4, 5]. Earlier resummation attempts include [6–11].

In this work we make use of a sophisticated resummation technique on de Sitter space known as stochastic inflation [12, 13]. Stochastic inflation provides a clever reorganization and resummation of perturbation theory in such a way that one obtains the correct leading order answers for infrared (super-Hubble) field correlators. The main observation due to Starobinsky [12] is that, while an interacting quantum field theory is essentially quantum on sub-Hubble length scales, it is classical on super-Hubble length scales if it couples non-conformally to gravity. Indeed, upon splitting the theory into short and long wavelength modes, one finds that the dynamics of long wavelength modes is particularly simple: the modes with a super-Hubble wavelength exhibit overdamped dynamics, due to Universe’s expansion gradient terms can be dropped and the only coupling between different modes comes from interactions. The coupling between the short and long wavelength modes can be modeled as a Markowian random force in the equation for the infrared modes. The resulting classical stochastic theory is particularly simple, and it can be shown to be equivalent to a Fokker-Planck equation for the single field probability distribution function (PDF) $\rho = \rho(\phi(\vec{x}), t)$, which is of the form,

$$\partial_t \rho = \frac{1}{3H} \partial_\phi (V' \rho) + \frac{H^3}{8\pi^2} \partial_\phi^2 \rho. \quad (1)$$

This equation is the stochastic equivalent of the von Neumann equation for the density operator in quantum field theory, and thus $\rho(\phi, t)$ can be thought of as the classical limit of the density operator.

Furthermore, it is known that a suitably adapted stochastic theory of inflation captures correctly field correlators in theories such as a self-interacting real scalar field theory [14], Yukawa theory [15] and scalar electrodynamics [16–18]. Due to the complex interplay between the constraints and dynamical field components in gravity, no consistent stochastic approximation has been so far developed for theories that include dynamical gravity. This is a pity, since this leaves us with an incomplete understanding of the dynamics of quantum fields on de Sitter space. The task is further complicated by the fact that only a few perturbative results are known that include quantum gravity [19–28]. The situation is much better

as regards perturbative results for various field theories on de Sitter [29–38]. Albeit useful, many of these results cannot be trusted at late times, when perturbative treatment breaks down. Therefore, one of the most burning unanswered questions of the physics of de Sitter space is:

What is the effective field theory that governs dynamics and probability distribution of the infrared fields in interacting field theories on de Sitter space?

That question is particularly interesting in the context of eternal inflation, as there one expects that any time dependence can be viewed as a dependence on some physical scale.

In order to make progress towards such an effective theory of inflation, note that – in the spirit of the renormalization group approach to effective field theories – the time dependence in (1) can be viewed as a (physical) scale dependence, where $\mu = \mu_0 e^{-Ht}$ (this dependence originates from the exponential expansion of physical scales in de Sitter space on super-Hubble length scales), such that $\partial_t \rightarrow -H\mu\partial_\mu$. With this observations Eq. (1) can be rewritten as a renormalization group (RG) equation,

$$\mu\partial_\mu\rho + \frac{1}{3H^2}\partial_\phi(V'\rho) + \frac{H^2}{8\pi^2}\partial_\phi^2\rho = 0. \quad (2)$$

This heuristic equation determines the PDF $\rho = \rho(\phi(\vec{x}), \mu)$ as a function of the field ϕ and the scale μ , which should be interpreted as the physical scale on which the field varies. An important question is how the PDF changes if one integrates out fluctuations above the scale μ . In the spirit of RG, one gets the effective PDF $\rho_{\text{eff}}(\phi, \mu) \equiv \rho_\mu(\phi)$ when fluctuations above that scale are integrated out. We shall postpone a concrete calculation of $\rho_\mu(\phi) \propto \exp\left(-\frac{8\pi^2}{3H^4}V_\mu(\phi)\right)$ for future work, and concentrate here on understanding the behavior of $\rho_\mu(\phi)$ and $V_\mu(\phi)$ in the limit when $\mu \rightarrow 0$. This PDF we refer to as the effective PDF, ρ_{eff} , and it signifies the probability distribution of (deep infrared) fields on the asymptotic timelike future infinity of de Sitter space. The program we propose here is illustrated on the Carter-Penrose diagrams in figure 1. The effective field theory (the effective potential and the corresponding effective probability distribution function) we discuss in this paper lives on the $\mu = 0$ ($t = \infty$) surface, and it is independent on the coordinates one uses. One can use this effective field theory to calculate the correlators of the fields that originate from (correlated) sub-Hubble vacuum fluctuations at some early time, and correspond to very large scale correlators of deep infrared fields at the asymptotic timelike future infinity of de Sitter. Albeit Eq. (2) is appealingly simple, it is not obtained by a rigorous derivation, and we shall address elsewhere the problem how to formally construct an effective field theory on de Sitter valid on some finite physical scale.

The effective theory approach advocated here becomes even better motivated when one recalls the observed equivalence between the zero mode partition function $Z_E(\phi_{0E})$ on Euclidean de Sitter space and

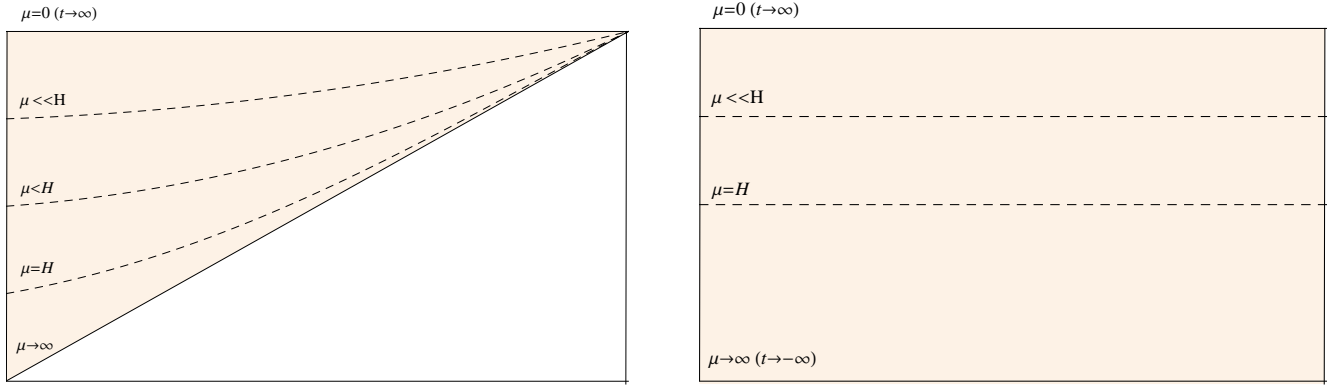


FIG. 1: The Carter-Penrose diagram of de Sitter space in flat coordinates used in this paper (left panel) and in global coordinates with positively curved spatial sections (right panel). The curves corresponding to constant physical scales $\mu \rightarrow \infty$, $\mu = H$, $\mu < H$, $\mu \ll H$ and $\mu = 0$ are also shown. Even though flat coordinates cover only 1/2 of de Sitter space (shaded), asymptotically the surface $\mu \rightarrow 0$ (or equivalently, the future timelike infinity surface i^+ , on which $t \rightarrow \infty$) is equal in both coordinates. We hence expect – and claim – that the effective theory that governs the field distribution on the surface $\mu = 0$ is independent on coordinates.

the PDF in stochastic approach at asymptotically late times $\rho(\phi, t \rightarrow \infty)$ [4],

$$Z_E(\phi_{0E}) = \rho(\phi, t \rightarrow \infty), \quad (3)$$

where Z_E denotes the Euclidean partition function, ϕ_{0E} is the zero field mode on Euclidean de Sitter space. [60] Namely, while the Euclidean space zero mode is distributed according to $Z_E[\phi_{0E}]$, the corresponding effective action, V_{Eff} , can be constructed by making use of a Legendre transformation. For a recent perturbative study of the Euclidean zero mode correlators see Ref. [5]. Inspired by this observation, an adaptation of this approach to stochastic inflation is the approach we advocate in this paper.

In order to make our study specific, we shall consider an $O(N)$ symmetric scalar field theory, whose tree level action is of the form,

$$S = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \sum_{a=1}^N (\partial_\mu \phi_a)(\partial_\nu \phi_a) - \frac{1}{2} m_0^2 \sum_{a=1}^N \phi_a^2 - \frac{\lambda_0}{4N} \left(\sum_{a=1}^N \phi_a^2 \right)^2 \right]. \quad (4)$$

where m_0 and λ_0 denote a bare field's mass and a bare quartic self-coupling, respectively, $g_{\mu\nu}$ is the metric tensor, $g^{\mu\nu}$ its inverse and $g = \det[g_{\mu\nu}]$. The metric signature we use is $(-, +, +, \dots)$. Upon a standard renormalization procedure [1], $m_0^2 \rightarrow m^2 \equiv -\mu^2$, and $\lambda_0 \rightarrow \lambda > 0$, where now m^2 and λ are finite renormalized parameters. The resulting action can be used as a starting point for stochastic theory. When $m^2 < 0$ ($\mu^2 > 0$) the theory exhibits symmetry breaking in Minkowski space, which is what we assume throughout this work. In the case when $N = 1$, one obtains the action of a real scalar field, and we shall

assume that the (renormalized) action is of the form,

$$S[\phi] = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right]. \quad (5)$$

When studied on Minkowski space, the vacuum of the theory (4) exhibits a scalar field condensate that breaks the $O(N)$ symmetry down to $O(N-1)$ (in the case when $N = 1$, $O(0)$ means that the symmetry $O(1) \equiv Z_2$ is completely broken by the condensate). This spontaneous symmetry breaking (SSB) results in a non-trivial vacuum $\mathcal{M} = O(N)/O(N-1)$. Excitations along these vacuum directions are massless, and they are known as Goldstone bosons. For example, when $N = 3$, the vacuum corresponds to the two dimensional sphere, $\mathcal{M} = O(3)/O(2) \sim S^2$, such that there are two massless Goldstone bosons, corresponding to excitations along the two orthogonal directions on S^2 . The vacuum of the theory is said to be trivial if $\vec{\phi}(\vec{x})$ maps all of the physical space R^3 into a point on field space, $\|\vec{\phi}\| = \phi_0 = \mu\sqrt{N/\lambda}$ (here $N = 3$). If however $\vec{\phi}(\vec{x})$ maps the 2-sphere $\|\vec{x}\| \rightarrow \infty$ of the physical space onto a 2-sphere of the internal space, the vacuum is said to be topologically nontrivial (the second homotopy group of the vacuum manifold is non-trivial, $\pi_2(\mathcal{M}) = Z$ [39]). This configuration is known as the global monopole, and once formed it is (topologically) stable. Global monopoles are an example of topological defects, which have been studied in the 1980s and 1990s as an alternative to inflation that can seed Universe's structure formation. When confronted with modern cosmic microwave background observations [40], these theories had to be abandoned. However, global defects have been invoked to drive inflation [41] or provide a possible explanation for dark energy [42].

The main purpose of this paper is to prove that no topological defects can survive in eternal inflation. In other words, deep infrared fields cannot see symmetry breaking and hence cannot form a condensate. For earlier work on this problem see Refs. [43–45]. The infrared effects from abundant particle production (in arbitrary number of space-time dimensions) in eternal inflation are so strong that they will eventually restore symmetry of any tree level potential of the form (4–5) and destroy any defects that might have formed on (sub-)Hubble scales [46–48]. As an important side remark, we note that our results suggest that very deep infrared modes will exhibit an enhanced non-Gaussianity of a non-perturbative character.

The paper is organized as follows. In section II we prove by explicit calculation that no condensate can form on de Sitter space in a theory (4) which at tree level exhibits a symmetry breaking ($m^2 < 0$). For pedagogical reasons, we present a separate proof for $O(1)$, $O(2)$ and the general $O(N)$ case. In section III we make use of the Legendre transform and show that the effective field theory at zero physical scale is convex for an arbitrary value of the background field. Again, we firstly study the $O(1)$ case, and then we prove convexity in the general $O(N)$ case. We also derive analytical approximations of the effective potential for small and large background field values. Finally, in section IV we summarize our main results,

and point at future research directions.

II. SYMMETRY RESTORATION

In order to prove that the symmetry in an $O(N)$ model is restored, it is sufficient to show that the curvature of the effective potential governing deep infrared field fluctuations is positive at the origin. We shall prove this by *reductio ad absurdum*. Namely, we shall show that the assumption that a condensate forms leads to a contradiction. We first consider a real scalar field, and then a complex (two component) scalar field and finally we discuss the general $O(N)$ symmetric case.

A. A real scalar field

Let us now consider a real scalar field, whose action is given in (5). We shall assume that the mass parameter $\mu^2 = -m^2 > 0$, such that the theory exhibits a spontaneous symmetry breaking in Minkowski space, and the field develops a condensate $\phi = \pm\phi_0 = \pm\mu\sqrt{6/\lambda}$ in its trivial vacuum state. In addition there are (topologically stable) domain wall solutions in which the field asymptotically condenses to $\phi = \pm\phi_0$ for $z \rightarrow \pm\infty$ (where z is some spatial direction). Domain walls could have formed in the early Universe by the Kibble mechanism [39], if at high temperatures the symmetry was restored by thermal field fluctuations. In this case, as the Universe expands it cools down and the model undergoes a phase transition.

We shall now consider the model (5) in inflationary (de Sitter) background. We shall argue that, while short scale field fluctuations can see a symmetry breaking potential, very long scale fluctuations see necessarily a potential in which the symmetry is restored. Let us begin our analysis by varying the action (5) with respect to ϕ . On super-Hubble the derivative terms can be dropped and the equation of motion for the mean (background) field ϕ_b simplifies to,

$$\left[-\mu^2 + \frac{\lambda}{2} \langle \delta\phi^2 \rangle_{\text{fin}} + \frac{\lambda}{6} \phi_b^2 \right] \phi_b = 0, \quad (6)$$

where the second term includes both the contribution from the fluctuations $\delta\phi = \phi - \phi_b$, that can be estimated by solving the Fokker-Planck equation (1) for stochastic inflation (while the sub-Hubble fluctuations contribute mainly to renormalize the coupling parameters, the super-Hubble field fluctuations are captured in $\langle \delta\phi^2 \rangle_{\text{fin}}$ and can be estimated by stochastic inflation, $\langle \delta\phi^2 \rangle_{\text{fin}} = \langle \delta\phi^2 \rangle_{\text{stoch}}$).

Ignoring the $\phi_b = 0$ solution (which is a local maximum), Eq. (6) implies,

$$\phi_b^2 = \phi_0^2 - 3 \langle \delta\phi^2 \rangle_{\text{stoch}} \geq 0. \quad (7)$$

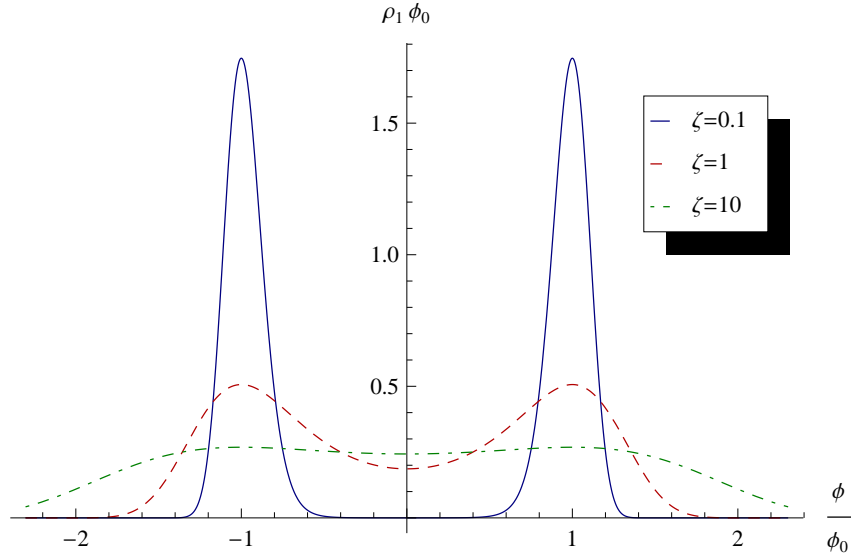


FIG. 2: The PDF $\phi_0 \times \rho_1(\phi)$ as a function of ϕ/ϕ_0 for the Hubble scale modes. The three curves represent $\zeta = 0.1$ (solid blue double peaked curve), $\zeta = 1$ (dashed red middle curve) and $\zeta = 10$ (dot-dashed green flattest curve), where $\zeta = 4\pi^2\mu^4/(\lambda H^4)$ is the dimensionless parameter characterizing the inverse coupling strength.

Assuming a stationary probability distribution $\rho(\phi, t) = \rho(\phi)$, Eq. (1) simplifies, and it is easily solved by

$$\begin{aligned} \rho_1(\phi) &= \frac{1}{Z_1} \exp\left(-\frac{8\pi^2}{3H^4}V_1\right); & Z_1(\zeta) &= \frac{\pi}{2}\phi_0 e^{\zeta/2} \left[I_{1/4}(\zeta/2) + I_{-1/4}(\zeta/2)\right] \\ V_1 &= -\frac{\mu^2}{2!}(\phi^2) + \frac{\lambda}{4!}(\phi^2)^2 \implies \frac{8\pi^2}{3H^4}V_1 = \zeta\left(\frac{\phi}{\phi_0}\right)^2 \left[\left(\frac{\phi}{\phi_0}\right)^2 - 2\right], \end{aligned} \quad (8)$$

where $\zeta = (4\pi^2\mu^4)/(\lambda H^4)$ denotes a dimensionless ‘inverse coupling’ parameter. The second way of writing V_1 in (8) is suggestive, as it indicates that – when written as a function of the rescaled field ϕ/ϕ_0 – all properties of the stationary ρ in (17) can be parametrized in terms of just one parameter: the inverse coupling parameter ζ . Thus in the limit when $\zeta \rightarrow \infty$ the theory is weakly coupled, while in the limit when $\zeta \rightarrow 0$ the theory becomes strongly coupled. While solution (8) has been known for quite a while [13], we give it a slightly different physical interpretation. For us Eq. (8) represents the PDF for the (stationary) modes that vary over approximately the Hubble scale. Albeit the PDFs that exhibit a nontrivial time dependence are interesting of their own right, for simplicity we do not study them here. Namely, Starobinsky and Yokoyama [13] have shown that, after a sufficient amount of time, every initial state necessarily reduces to the PDF in (8), making the stationary distribution (8) an attractor.

In figure 2 we show the initial PDF for these stationary (mainly) Hubble scale modes. We shall refer it to as the PDF for Hubble scale modes, and the corresponding $V = V(\phi, \mu \sim H)$ is the effective (Hubble scale) potential. Of course, this potential still exhibits symmetry breaking, which can be seen from the characteristic double peak structure of the PDF in figure 2, which is more pronounced for strong couplings

($\zeta \ll 1$). Nevertheless, as we show below, the inclusion of these fluctuations into (7) is sufficient to prevent background field condensation. Indeed, inserting

$$\langle \delta \phi^2 \rangle_{\text{stoch}} = \langle \phi^2 \rangle_{\text{stoch}} + \phi_{\text{b}}^2$$

into Eq. (7) results in

$$\phi_{\text{b}}^2 = \frac{\phi_0^2}{4} (1 - 3 \langle \phi^2 / \phi_0^2 \rangle_{\text{stoch}}) \geq 0 \quad (9)$$

$$\left\langle \frac{\phi^2}{\phi_0^2} \right\rangle_{\text{stoch}} = \frac{1}{2} \left(1 + \frac{I_{3/4}(\zeta/2) + I_{-3/4}(\zeta/2)}{I_{1/4}(\zeta/2) + I_{-1/4}(\zeta/2)} \right), \quad (10)$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \nu + 1)k!} \left(\frac{z}{2} \right)^{2k+\nu} \quad (11)$$

is the series representation around $z = 0$ for the modified Bessel's function of the first kind. When (10) is inserted into Eq. (9) one gets,

$$\phi_{\text{b}}^2 = -\frac{\phi_0^2}{8} \left(1 + 3 \frac{I_{-3/4}(\zeta/2) + I_{3/4}(\zeta/2)}{I_{-1/4}(\zeta/2) + I_{1/4}(\zeta/2)} \right) \geq 0, \quad (12)$$

which is impossible to satisfy, thus completing the proof. The proof is an immediate consequence of the observation that, if $\nu > -1$, $I_\nu(z)$ is strictly positive for all $z > 0$. This is true because for $\nu > -1$ all coefficients in the series (11) for $I_\nu(z)$ are positive when $\nu > -1$, and this condition is satisfied for all Bessel's functions occurring in (12). We have thus proved that, when the field fluctuations evaluated within stochastic theory of inflation are taken account of, no phase transition in the $O(1)$ model (5) can occur, *i.e.* there can be no field condensate.

B. The $O(N)$ model

The proof in the $O(N)$ symmetric model (4) is analogous to the proof in the $O(1)$ case. The main complication is the enlarged symmetry, and we shall devote some attention to explain the resulting differences. In the $O(N)$ case the field $\vec{\phi} = (\phi_a)$ ($a = 1, \dots, N$) is a N -component vector, and when a mean field condensate is present, a suitable orthogonal rotation can bring the condensate direction into the direction of ϕ_1 , such that $\phi_{\text{b}} = \phi_{1\text{b}}$ and $(\phi_0)^2 = (\phi_1)_0^2 = N\mu^2/\lambda$. The generalization of (6) to the $O(N)$ case is (*cf.* Ref. [1])

$$\phi_{\text{b}}^2 = (\phi_1)^2 = \phi_0^2 - 3 \langle (\delta \phi_1)^2 \rangle_{\text{stoch}} - (N-1) \langle (\delta \phi_2)^2 \rangle_{\text{stoch}} \geq 0, \quad (13)$$

where

$$\langle(\delta\phi_1)^2\rangle = \langle(\phi_1 - \phi_b)^2\rangle = (\phi_b)^2 + \langle(\phi_1)^2\rangle = (\phi_b)^2 + \frac{1}{N}\langle(\vec{\phi})^2\rangle \quad (14)$$

$$\langle(\delta\phi_2)^2\rangle = \langle(\phi_2)^2\rangle = \frac{1}{N}\langle(\vec{\phi})^2\rangle. \quad (15)$$

Because the scalar potential in (4) depends only on $\sum_{a=1}^N(\phi_a)^2 = (\vec{\phi})^2$, the linear term in (14) does not contribute, and $\langle(\delta\phi_i)^2\rangle = \langle(\delta\phi_j)^2\rangle$ ($\forall i, j \in \{1, \dots, N\}$). Inserting Eqs. (14–15) into (13) results in

$$\phi_b^2 = \frac{\phi_0^2}{4} \left(1 - \frac{N+2}{N} \langle(\vec{\phi}/\phi_0)^2\rangle_{\text{stoch}} \right) \geq 0. \quad (16)$$

We shall now show that this inequality has no non-trivial solution, *i.e.* that no condensate can form for stationary solutions in stochastic inflation.

But, before we consider the general case, it is instructive to consider the $O(2)$ model (complex scalar field), since in this case the results are particularly simple. We have

$$\begin{aligned} \rho(\phi_1, \phi_2) &= \frac{1}{Z_2} \exp\left(-\frac{8\pi^2}{3H^4} V_2\right), \quad \frac{1}{Z_2} = \frac{1}{\phi_0^2} \frac{2}{\pi^{3/2}} \frac{\sqrt{\zeta} e^{-\zeta}}{1 + \text{erf}(\sqrt{\zeta})}, \quad \zeta = \frac{4\pi^2 \mu^4}{3\lambda H^4} \\ V_2 &= -\frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{8}(\phi_1^2 + \phi_2^2)^2 \Rightarrow \frac{8\pi^2}{3H^4} V_2 = \zeta(\vec{\phi}/\phi_0)^2 [(\vec{\phi}/\phi_0)^2 - 2], \end{aligned} \quad (17)$$

where $\text{erf}(z) = (2/\sqrt{\pi}) \int_0^z dt e^{-t^2}$ denotes the *error* function and we normalized ρ by demanding $\int_{-\infty}^{\infty} d\phi_1 d\phi_2 \rho = 1$.

In order to make progress on inequality (16), we need to calculate $\langle\vec{\phi}^2\rangle_{\text{stoch}}$. Making use of (17) for $N = 2$ we immediately get,

$$\begin{aligned} \langle\vec{\phi}^2\rangle_{\text{stoch}} &= \frac{\pi\phi_0^4}{Z_2} \int_0^\infty d\varphi^2 \varphi^2 \exp\{-\zeta[(\varphi^2 - 1)^2 - 1]\} \\ &= 2\phi_0^2 \frac{\sqrt{\zeta/\pi}}{1 + \text{erf}(\sqrt{\zeta})} \int_{-1}^\infty du (u+1) e^{-\zeta u^2} \\ &= \phi_0^2 \left(1 + \frac{e^{-\zeta}}{\sqrt{\pi\zeta}[1 + \text{erf}(\sqrt{\zeta})]} \right), \end{aligned} \quad (18)$$

where $\varphi^2 = (\phi_1^2 + \phi_2^2)/\phi_0^2$ and $u = \varphi^2 - 1$. Upon inserting this result into (16) with $N = 2$ yields

$$\phi_b^2 = -\frac{\phi_0^2}{4} \left[1 + \frac{2e^{-\zeta}}{\sqrt{\pi\zeta}[1 + \text{erf}(\sqrt{\zeta})]} \right] > 0. \quad (19)$$

Now, since $\text{erf}(z) > 0$ ($\forall z > 0$) Eq. (19) cannot be satisfied for any $\zeta > 0$, proving the impossibility of condensate formation for the $O(2)$ case.

We are now ready to consider symmetry breaking in the general $O(N)$ symmetric case. In this case the PDF for a stationary state is the following generalization of Eqs. (17)

$$\begin{aligned}
\rho &= \frac{1}{Z} \exp \left(-\frac{8\pi^2}{3H^4} V \right), & \zeta &= \frac{2N\pi^2\mu^4}{3\lambda H^4}, & \phi_0^2 &= \frac{N\mu^2}{\lambda}, \\
Z &= \phi_0^N \frac{\Omega(S^{N-1})}{4\zeta^{N/4}} \left[\Gamma\left(\frac{N}{4}\right) \times {}_1F_1\left(\frac{N}{4}; \frac{1}{2}; \zeta\right) + 2\sqrt{\zeta}\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{3}{2}; \zeta\right) \right] \\
V &= -\frac{\mu^2}{2} \left(\sum_{a=1}^N \phi_a^2 \right) + \frac{\lambda}{4N} \left(\sum_{a=1}^N \phi_a^2 \right)^2 \Rightarrow \frac{8\pi^2}{3H^4} V = \zeta \left(\frac{\vec{\phi}}{\phi_0} \right)^2 \left[\left(\frac{\vec{\phi}}{\phi_0} \right)^2 - 2 \right],
\end{aligned} \tag{20}$$

where $\Omega(S^{N-1}) = 2\pi^{N/2}/\Gamma(N/2)$ denotes the volume (surface area) of the unit $N-1$ dimensional sphere.

The two point function is then:

$$\begin{aligned}
\langle \vec{\phi}^2 \rangle_{\text{stoch}} &= \Omega(S^{N-1}) \frac{\phi_0^{N+2}}{Z} \int_0^\infty d\varphi \varphi^{N+1} e^{-\zeta[(\varphi^2-1)^2-1]} \\
&= \Omega(S^{N-1}) \frac{\phi_0^{N+2}}{Z} \int_{-1}^\infty du (u+1)^{N/2} e^{-\zeta u^2} e^\zeta \\
&= \frac{\phi_0^2}{\sqrt{\zeta}} \times \frac{\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{1}{2}; \zeta\right) + 2\sqrt{\zeta}\Gamma\left(\frac{N}{4}+1\right) \times {}_1F_1\left(\frac{N}{4}+1; \frac{3}{2}; \zeta\right)}{\Gamma\left(\frac{N}{4}\right) \times {}_1F_1\left(\frac{N}{4}; \frac{1}{2}; \zeta\right) + 2\sqrt{\zeta}\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{3}{2}; \zeta\right)},
\end{aligned} \tag{21}$$

where $\varphi^2 = \vec{\phi}^2/\phi_0^2$ and $u = \varphi^2 - 1$. Upon inserting (21) into (16) we get

$$\phi_b^2 = \frac{\phi_0^2}{4} F(\zeta, N) \tag{22}$$

where

$$F(\zeta, N) = 1 - \frac{N+2}{N} \frac{2\Gamma\left(\frac{N}{4}+1\right) \times {}_1F_1\left(\frac{N}{4}+1; \frac{3}{2}; \zeta\right) + \frac{1}{\sqrt{\zeta}}\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{1}{2}; \zeta\right)}{\Gamma\left(\frac{N}{4}\right) \times {}_1F_1\left(\frac{N}{4}; \frac{1}{2}; \zeta\right) + 2\sqrt{\zeta}\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{3}{2}; \zeta\right)}. \tag{23}$$

In the appendix we show that $F(\zeta, N) < 0$ for any integer $N > 0$ and $\zeta > 0$, which completes the proof that there can be no mean field condensate in an $O(N)$ symmetric scalar field theory in de Sitter space.

III. EFFECTIVE POTENTIAL

While in section II we show that inflationary fluctuations treated within the stochastic formalism are strong enough to prevent formation of a scalar condensate in any $O(N)$ symmetric scalar theory in de Sitter inflation, here we prove a much more powerful theorem. Namely, we show that the effective potential governing the PDF of very long wave length fluctuations in inflation is strictly convex (for arbitrary value of the field). This represents a general proof that deep infrared modes in de Sitter space see a symmetry restoring potential, and hence their PDF must be peaked at zero field value. Inspired by the observation that the partition function of the zero Euclidean mode equals the PDF for time independent field configurations (3) we shall use the effective action formalism to construct V_{eff} and the corresponding ρ_{eff} . We first consider the $O(1)$ case, and then subsequently the general $O(N)$ symmetric model.

A. The $O(1)$ model

Since $\rho(\phi)$ yields a PDF for ϕ that varies on the Hubble scale, it is reasonable to posit that ρ_{eff} obtained by the conventional field theoretic technique of Legendre transform [49] will give a PDF for deep infrared ($\mu \rightarrow 0$) scalar field fluctuations ϕ_b , whereby all higher energy fluctuations have been integrated out. Since any field that originates from sub-Hubble scale fluctuations rapidly redshifts during inflation, the effective potential that we derive here yields the field distribution at the asymptotically late times ($t \rightarrow \infty$), at the asymptotic future timelike infinity i^+ of de Sitter space, as illustrated in figure 1.

Adding a source current $J(\vec{x})$ to $V(\phi)$ one can define a partition function (at a point \vec{x}) $Z(J(\vec{x}))$ as

$$Z(J) \equiv e^{-W(J)} = \int d\phi \rho_1(\phi) e^{J\phi} = \langle e^{J\phi} \rangle. \quad (24)$$

The effective potential V_{eff} is then given as a Legendre transform of $W(J) = -\ln[Z(J)]$,

$$\frac{8\pi^2}{3H^4} V_{\text{eff}}(\phi_b) = W(J) + J\phi_b; \quad \phi_b = \frac{\partial \ln Z(J)}{\partial J} \quad (25)$$

and the corresponding PDF is then

$$\rho_{\text{eff}}(\phi_b) = \frac{1}{Z_{\text{eff}}} \exp\left(-\frac{8\pi^2}{3H^4} V_{\text{eff}}(\phi_b)\right); \quad Z_{\text{eff}} = \int d\phi_b \exp\left(-\frac{8\pi^2}{3H^4} V_{\text{eff}}\right). \quad (26)$$

In order to study convexity of $V_{\text{eff}}(\phi_b)$ we shall make use of the equation of motion for ϕ_b and its derivative,

$$\frac{\partial V_{\text{eff}}(\phi_b)}{\partial \phi_b} = J; \quad \frac{\partial^2 V_{\text{eff}}(\phi_b)}{\partial \phi_b^2} = \frac{\partial J(\phi_b)}{\partial \phi_b}. \quad (27)$$

V_{eff} is convex if $\partial^2 V_{\text{eff}} / \partial \phi_b^2 > 0$ for all ϕ_b . The inverse of this must also be positive,

$$\left(\frac{\partial^2 V_{\text{eff}}(\phi_b)}{\partial \phi_b^2}\right)^{-1} = \frac{\partial^2 \ln[Z(J)]}{\partial J^2} = \frac{Z''(J)}{Z(J)} - \left(\frac{Z'(J)}{Z(J)}\right)^2 > 0. \quad (28)$$

Now, from the definition of $Z(J)$ in Eq. (24) it follows

$$Z(J) = \sum_{n=0}^{\infty} \frac{\langle \phi^n \rangle J^n}{n!} = \langle e^{J\phi} \rangle, \quad (29)$$

where

$$\left\langle \frac{\phi^{2n}}{\phi_0^{2n}} \right\rangle_{\text{stoch}} = \frac{\Gamma\left(\frac{n}{2} + \frac{1}{4}\right) \times {}_1F_1\left(\frac{n}{2} + \frac{1}{4}; \frac{1}{2}; \zeta\right) + 2\sqrt{\zeta} \Gamma\left(\frac{n}{2} + \frac{3}{4}\right) \times {}_1F_1\left(\frac{n}{2} + \frac{3}{4}; \frac{3}{2}; \zeta\right)}{\pi e^{\zeta/2} \zeta^{\frac{n}{2} + \frac{1}{4}} (I_{1/4}(\zeta/2) + I_{-1/4}(\zeta/2))} \quad (30)$$

and $\langle \phi^{2n+1} \rangle_{\text{stoch}} = 0$ ($n = 0, 1, 2, \dots$). Furthermore, we have

$$Z'(J) = \langle \phi e^{J\phi} \rangle = Z(J) \langle \phi \rangle_J; \quad Z''(J) = \langle \phi^2 e^{J\phi} \rangle = Z(J) \langle \phi^2 \rangle_J, \quad (31)$$

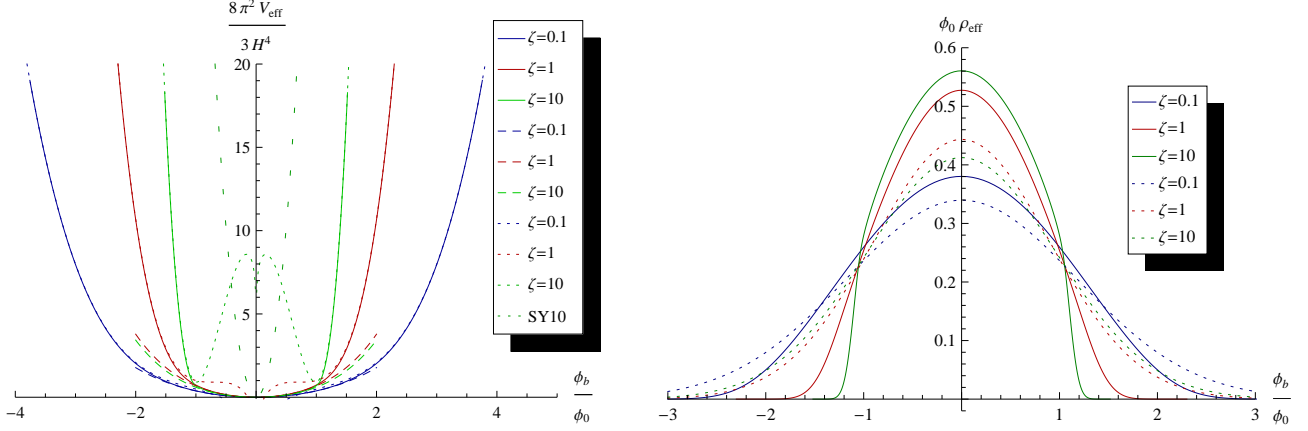


FIG. 3: *Left panel:* V_{eff} as a function of ϕ_b for $\zeta = 0.1, 1, 10$ (solid blue innermost curve, solid red middle curve and solid green outermost curve). We also show the weak field analytic effective potential (39) (dotted curves), the analytic expression (43) for large ϕ_b (dashed curves), and finally the Starobinsky-Yokoyama effective potential (38) (sparse green dashed innermost curve) just for $\zeta = 10$ (in order not to overcrowd the plot). *Right panel:* ρ_{eff} as function of ϕ_b for $\zeta = 0.1, 1, 10$ (solid curves, the same color code as on the left panel) and the best Gaussian fits close to the origin (dotted curves).

where $\langle \cdot \rangle_J$ denotes averaging with respect to $\rho_J(\phi) = \rho(\phi)e^{J\phi}$, which is also positive. With this in mind we can rewrite the convexity condition (28) as

$$\left(\frac{\partial^2 V_{\text{eff}}(\phi_b)}{\partial \phi_b^2} \right)^{-1} = \langle \phi^2 \rangle_J - \langle \phi \rangle_J^2 > 0. \quad (32)$$

Now, since $\rho_J(\phi)$ is positive definite, we can make use of the Cauchy-Schwarz theorem for probability theory, [61] which for $f = 1$ and $g = \phi$ (see footnote [61]) reads

$$\langle \phi \rangle_J^2 < \langle \phi^2 \rangle_J, \quad (33)$$

where the strong inequality follows because 1 and ϕ are linearly independent functions. This immediately implies that the (strong) convexity condition (32) for V_{eff} is satisfied, completing the proof.

B. Analytical approximations for V_{eff}

In figure 3 we show numerical plots for V_{eff} and ρ_{eff} as a function of the deep infrared background field ϕ_b , for different values of the dimensionless inverse coupling parameter: $\zeta = 0.1, 1, 10$. The figures do not represent exact V_{eff} , but instead they are based on taking a finite number of terms in the sum (29) involved in the definition of $Z(J)$. However, we have checked that increasing the number of terms in the sum does not change the form of V_{eff} to such an extent to be visible on the plots. It is apparent that, for small

values of ϕ_b , V_{eff} exhibits a simple quadratic dependence on ϕ_b , implying an approximately Gaussian ρ_{eff} (that can be characterized by a positive mass term), as can be seen on the right panel of figure 3. In the large field limit when $\phi_b \gg \phi_0$ the effective potential is, as expected, quartic, $[(8\pi^2/(3H^4))V_{\text{eff}}] \sim \zeta \phi_b^4$. However there is no good smooth matching between the quadratic and quartic behavior, and around $\phi_b \sim \phi_0$ the quadratic behavior turns quickly into a quartic behavior as one would expect from a Maxwell construction. Indeed, forcing a quadratic plus quartic fit onto V_{eff} when $\zeta \gg 1$ results in a fit with a double well structure. These observations then imply that weakly coupled theories exhibit large deviations from the behavior expected based on the Starobinsky-Yokoyama procedure, in the sense that small amplitude fluctuations of deep infrared fields are enhanced when compared with the Starobinsky-Yokoyama result discussed in section II and below.

In fact, it is worth making a detailed comparison of the mass implied by the effective action in figure 3 and that implied by the Starobinsky-Yokoyama procedure [5, 13] which, when generalized to a finite background field ϕ_b , yields,

$$m_{\text{SY}}^2(\phi_b) = -\mu^2 + \frac{\lambda}{2} \langle (\phi - \phi_b)^2 \rangle_{\text{stoch}} = \mu^2 \left[3 \left(\left\langle \frac{\phi^2}{\phi_0^2} \right\rangle_{\text{stoch}} + \frac{\phi_b^2}{\phi_0^2} \right) - 1 \right], \quad (34)$$

where $\langle (\phi/\phi_0)^2 \rangle_{\text{stoch}}$ is given in Eq. (10). On the other hand, the (numerical) mass that measures the curvature of the effective potential in figure 3 is given by

$$M^2(\phi_b) = \frac{\partial^2 \left[\frac{8\pi^2}{3H^4} V_{\text{eff}}(\phi_b) \right]}{\partial (\phi_b/\phi_0)^2}. \quad (35)$$

Now, one can easily show that expressions (10) and (35) are related as,

$$m^2(\phi_b) = \frac{\mu^2}{4\zeta} M^2(\phi_b), \quad (36)$$

which – if the Starobinsky-Yokoyama prescription is correct – implies the following mass term,

$$M_{\text{SY}}^2(\phi_b) = 12\zeta \left[\left\langle \frac{\phi^2}{\phi_0^2} \right\rangle_{\text{stoch}} + \frac{\phi_b^2}{\phi_0^2} - \frac{1}{3} \right]. \quad (37)$$

The corresponding effective potential is then

$$\frac{8\pi^2}{3H^4} V_{\text{eff SY}}(\phi_b) = 6\zeta \left[\left\langle \frac{\phi^2}{\phi_0^2} \right\rangle_{\text{stoch}} - \frac{1}{3} \right] \frac{\phi_b^2}{\phi_0^2} + \zeta \left(\frac{\phi_b}{\phi_0} \right)^4 + \frac{8\pi^2}{3H^4} V_0, \quad (38)$$

where V_0 is an integration constant (a linear term in ϕ_b is forbidden by symmetry).

On the other hand, one can calculate the effective potential around $\phi_b = 0$ by making use of Eqs. (24–25). Upon expanding $e^{J\phi}$ in (24) in powers of $J\phi$ and keeping the terms up to $(J\phi)^4$ we get

$$\frac{8\pi^2}{3H^4} V_{\text{eff}} = \frac{1}{2} \frac{\phi_b^2}{\langle \phi^2 \rangle_{\text{stoch}}} + \frac{1}{8} \left(1 - \frac{1}{3} \frac{\langle \phi^4 \rangle_{\text{stoch}}}{\langle \phi^2 \rangle_{\text{stoch}}^2} \right) \frac{\phi_b^4}{\langle \phi^2 \rangle_{\text{stoch}}^2} + \mathcal{O}((\phi_b/\phi_0)^6), \quad (39)$$

which yields for the mass around $\phi_b = 0$,

$$M^2(0) = \frac{\phi_0^2}{\langle \phi^2 \rangle_{\text{stoch}}}, \quad (40)$$

implying the following physical mass (36) at $\phi_b = 0$

$$m^2(0) = \frac{3H^4}{8\pi^2 \langle \phi^2 \rangle_{\text{stoch}}}. \quad (41)$$

There is a very intriguing connection of the result (41), with the mean field (one loop) result for the mass [1]. Recall that in $D = 4$, for a field of the mass parameter m^2 , the mean field treatment gives $(\lambda/2)\imath\Delta(x; x)$, where the coincident propagator is, $\imath\Delta(x; x) = (3H^4)/(8\pi^2 m^2)$. Now, assume that the mass m is entirely created by stochastic fluctuations, thus $m^2 \rightarrow (\lambda/2)\langle \phi^2 \rangle_{\text{stoch}}$, the one-loop mean field formula (with a stochastic flavor) then gives $m^2(0) = (3H^4)/(8\pi^2 \langle \phi^2 \rangle_{\text{stoch}})$, which is precisely the result (41). Of course, this is a very heuristic ‘derivation’, and it is very difficult to imagine that one could arrive at that result by any other but the rigorous method advocated in this work.

At a first sight the mass (40–41) may appear bizarre, as it exhibits a very different dependence on the coupling parameter $\zeta \propto 1/\lambda$ from that implied by the Starobinsky-Yokoyama formula (37). At a second sight however (41) is not surprising at all; it simply tells us that for small field amplitudes,

$$\rho_{\text{eff}} \propto \exp\left(-\frac{\phi_b^2}{2\langle \phi^2 \rangle_{\text{stoch}}}\right); \quad (\phi_b \ll \phi_0),$$

which is precisely what one would expect from a stochastic theory in its Gaussian limit! (Albeit the theory is far from being Gaussian, for weak fields it is approximately Gaussian.)

Let us now check whether our result (40) agrees with the numerical findings in figure 3. In the weak coupling limit when $\zeta \rightarrow \infty$, $M^2(0) \rightarrow 1$, while in the strong coupling limit ($\zeta \rightarrow 0$), $M^2(0) \rightarrow [\Gamma(1/4)/\Gamma(3/4)]\sqrt{\zeta}$. For $\zeta = 0.1, 1, 10$, Eq. (40) predicts $\{0.72, 1.20, 1.03\}$, which agrees well with the approximate curves on the left panel of figure 3. These values are to be compared with the values implied by Eq. (37): $M_{\text{SY}}^2(0) \simeq \{1.3, 6, 77\}$ for $\zeta = \{0.1, 1, 10\}$. We have thus found that the Starobinsky-Yokoyama procedure typically predicts a much larger mass term at the origin than what is implied by the effective potential approach.

In order to get a better feeling for the coupling dependence of the physical mass (41), consider the weak and strong coupling regimes of Eq. (41),

$$m^2(0) \xrightarrow{\lambda \rightarrow 0} \frac{\lambda H^4}{16\pi^2 \mu^2}; \quad m^2(0) \xrightarrow{\lambda \rightarrow \infty} \frac{\Gamma(1/4)}{\Gamma(3/4)} \frac{\sqrt{\lambda} H^2}{8\pi} \simeq 0.1177 \sqrt{\lambda} H^2. \quad (42)$$

Thus, the strong coupling behavior is qualitatively the same as that of Starobinsky-Yokoyama [13] $m^2(0) \sim \sqrt{\lambda} H^2$ (albeit the dimensionless prefactors do not agree), but in the weak coupling regime the results are

qualitatively different. The latter result in Eq. (42) should not surprise us: in the strong coupling regime the curvature around the origin ($-\mu^2$) is negligible, and one recovers the expected result $m^2(0) \sim \sqrt{\lambda}H^2$ (which is the loop counting parameter of the resummed perturbation theory). On the other hand, the weak coupling result in (42) is quite surprising, in that it tells us that one almost recovers the naïve dependence on the coupling constant, $m^2 \propto \lambda H^2$ (the additional factor H^2/μ^2 is not easily explicable). What is interesting about that result is that, in the limit when $\lambda \rightarrow 0$ and/or $\mu \rightarrow \infty$, small amplitude fluctuations at the timelike asymptotic infinity of de Sitter behave as massless field fluctuations, with the caveat that the amplitude of these fluctuations must satisfy $|\phi_b| < \phi_0 = \mu\sqrt{6/\lambda}$, which is the scale at which the ϕ_b^4 term in the effective potential kicks in.

Next we consider the large field behavior of the effective potential (25). In the Appendix we show that the asymptotic effective potential is of the form,

$$\begin{aligned} \frac{8\pi^2}{3H^4}V_{\text{eff}}(\phi_b) &= \zeta \left[\left(\frac{\phi_b}{\phi_0} \right)^4 - 2 \left(\frac{\phi_b}{\phi_0} \right)^2 \right] + \frac{8\pi^2}{3H^4}\Delta V_{\text{eff}}(\phi_b) \\ \frac{8\pi^2}{3H^4}\Delta V_{\text{eff}}(\phi_b) &\approx \frac{\zeta}{2} + \frac{1}{2} \ln \left(\frac{3\pi\zeta\phi_b^2}{2\phi_0^2} \right) + \ln \left(I_{\frac{1}{4}} \left(\frac{\zeta}{2} \right) + I_{-\frac{1}{4}} \left(\frac{\zeta}{2} \right) \right), \quad (\phi_b \gg \phi_0), \end{aligned} \quad (43)$$

or equivalently,

$$\begin{aligned} V_{\text{eff}}(\phi_b) &= -\frac{\mu^2}{2}\phi_b^2 + \frac{\lambda}{4!}\phi_b^4 + \Delta V_{\text{eff}}(\phi_b) \\ \Delta V_{\text{eff}}(\phi_b) &\approx \frac{3\mu^4}{4\lambda} + \frac{3H^4}{16\pi^2} \ln \left(\frac{\phi_b^2}{H^2} \right) + \frac{3H^4}{8\pi^2} \ln \left[\frac{\pi^{3/2}\mu}{H} \left(I_{\frac{1}{4}} \left(\frac{\zeta}{2} \right) + I_{-\frac{1}{4}} \left(\frac{\zeta}{2} \right) \right) \right], \quad (\phi_b \gg \phi_0), \end{aligned} \quad (44)$$

Firstly, the potential (43–44) fits excellently the numerical V_{eff} for large ϕ_b , as can be seen in the left panel of figure 3 (dotted curves). Secondly, the leading term $(\zeta(\phi_b/\phi_0)^4)$ and the subleading term $(-2\zeta(\phi_b/\phi_0)^2)$ are the same as in the tree level potential (8). However, there is also the contribution ΔV_{eff} that originates from integrating out fluctuations on all super-Hubble scales, and it consists of a logarithmic contribution $((1/2)\ln[(\phi_b/\phi_0)^2])$ and a field independent contribution. These contributions are felt by large field excitations as a result of integrating out fluctuations on all scales, and they introduce an average upward shift in the effective potential that grows weakly (logarithmically) with the field amplitude. This means that large amplitude, deep infrared fields in de Sitter are not distributed according to the Coleman-Weinberg effective potential [50]. A comparison with the Starobinsky-Yokoyama effective potential (38) reveals that, in the large amplitude limit, it recovers the leading (quartic) contribution in the field correctly, but it fails at lower orders (quadratic, *etc*), and hence it does not represent a very good fit, as can be seen on the left panel in figure 3 (for $\zeta = 10$ long dashed green). In order to get a better feeling for ΔV_{eff} in (44) notice that in the weak and strong coupling case it reduces to,

$$\Delta V_{\text{eff}}(\phi_b) \xrightarrow{\lambda \rightarrow 0} \frac{3\mu^4}{2\lambda} + \frac{3H^4}{16\pi^2} \ln \left(\frac{\lambda\phi_b^2}{\mu^2} \right); \quad \Delta V_{\text{eff}}(\phi_b) \xrightarrow{\lambda \rightarrow \infty} \frac{3H^4}{16\pi^2} \ln \left(\frac{\pi^2\sqrt{\lambda}\phi_b^2}{\Gamma^2(3/4)H^2} \right). \quad (45)$$

This means that the contribution from integrated fluctuations grows as λ decreases. In fact there is an upper bound for ΔV_{eff} which comes from the natural assumption, $\mu, \phi_0 \lesssim m_{\text{p}}$, which then implies that $\Delta V_{\text{eff}} \lesssim m_{\text{p}}^4/4$, which is of the order of the Planck energy density. In the strong coupling regime however, ΔV_{eff} remains limited to $\Delta V_{\text{eff}} \sim H^4$.

Next, we consider the backreaction on the geometry, for which we need the energy stored in the fluctuations. According to the Starobinsky-Yokoyama prescription, the energy density is stored in quantum fluctuations is

$$\langle V \rangle_{\text{stoch}} = -\frac{\mu^2}{2} \langle \phi^2 \rangle_{\text{stoch}} + \frac{\lambda}{4!} \langle \phi^4 \rangle_{\text{stoch}}, \quad (46)$$

where $\langle \phi^2 \rangle_{\text{stoch}}$ and $\langle \phi^4 \rangle_{\text{stoch}}$ are given in Eqs. (10) and (30). For $\zeta = \{0.1, 1, 10\}$ Eq. (46) gives $[8\pi^2/(3H^4)]\langle V \rangle_{\text{stoch}} = \{0.11, -0.58, -9.48\}$. On the other hand, we can numerically evaluate $\langle V_{\text{eff}} \rangle$ and we get for $\zeta = \{0.1, 1, 10\}$, $[8\pi^2/(3H^4)]\langle V_{\text{eff}} \rangle_{\text{stoch}} = \{0.39, 0.34, 0.23\}$, respectively. We thus see that the naïve contribution from quantum stochastic fluctuations as given by the Starobinsky-Yokoyama procedure starts as a positive constant ($= 1/4$) when $\zeta \rightarrow 0$, but then it becomes negative as ζ increases, approaching $[8\pi^2/(3H^4)]\langle V \rangle_{\text{stoch}} \simeq -2\zeta$ when $\zeta \rightarrow \infty$. This is the energy density perceived by the fields that vary over the Hubble scale, but cannot be used for backreaction on the background geometry, for which we need the energy density perceived by very deep infrared modes, which is just $\langle V_{\text{eff}} \rangle_{\text{stoch}}$. The contribution $[8\pi^2/(3H^4)]\langle V_{\text{eff}} \rangle_{\text{stoch}}$ is (a) positive for all values of the coupling constants ζ and (b) it decreases as ζ increases. This is so because, when $\lambda \rightarrow 0$ and close to the origin where ρ is significantly different from zero, $V_{\text{eff}} \simeq 0$ and it is very flat, and where V_{eff} is large ($\phi_{\text{b}} \gg \phi_0$), ρ shoots rapidly to zero. The energy $\langle V_{\text{eff}} \rangle_{\text{stoch}}$ is to be added as a *positive backreaction* $(8\pi G_N)\langle V_{\text{eff}} \rangle_{\text{stoch}}$ to the background cosmological constant Λ_0 , effectively increasing the rate of Universe's expansion. Since $[8\pi^2/(3H^4)]\langle V_{\text{eff}} \rangle_{\text{stoch}}$ is not greater than unity, the contribution to the cosmological constant is of the order $\Delta\Lambda = (8\pi G_N)\langle V_{\text{eff}} \rangle_{\text{stoch}} = (8\pi G_N)\mathcal{O}(H^4)$, which implies $\Delta\Lambda/\Lambda_0 = \mathcal{O}(H^2/m_{\text{p}}^2)$ and hence it is typically small.

Next, we shall consider the entropy stored in fluctuations at a point \vec{x} at asymptotic timelike future infinity i^+ . Since stochastic formalism does not contain all of the information necessary to reconstruct the von Neumann entropy of the state [51–54], we resort to the less fundamental, but simpler, concept of the Shannon (or Gibbs) entropy [51], which is defined as $S_{\text{Shannon}} = -\langle \ln(\rho\mu_0) \rangle_{\text{stoch}}$, which when adapted to our problem yields,

$$S_{\text{Shannon,SY}} = \frac{8\pi^2}{3H^4} \langle V \rangle_{\text{stoch}} + \ln\left(\frac{Z_1}{\mu_0}\right), \quad (47)$$

which in the strong and weak coupling limit gives (for simplicity we set the scale $\mu_0 = \phi_0$),

$$S_{\text{Shannon,SY}} \xrightarrow{\zeta \rightarrow 0} \ln\left(\frac{\zeta}{4}\right) + \frac{1}{4} + \ln\left(\frac{\pi}{\sqrt{2}\Gamma(3/4)}\right); \quad S_{\text{Shannon,SY}} \xrightarrow{\zeta \rightarrow \infty} -\frac{1}{2} \ln\left(\frac{\zeta}{\pi}\right).$$

The fact that in the weakly coupled regime $S_{\text{Shannon},\text{SY}} < 0$ should not be of a concern, since $S_{\text{Shannon},\text{SY}}$ in (47) is defined up to a constant determined by a mass scale μ_0 , and μ_0 can be chosen such to keep $S_{\text{Shannon},\text{SY}} > 0$. In particular, for $\zeta = \{0.1, 1, 10\}$ one gets, $S_{\text{Shannon},\text{SY}} = \{1.52, 1.09, -0.033\}$. This is to be compared with the Shannon entropy implied by the effective theory, $S_{\text{Shannon},\text{eff}} = -\langle \ln(\rho_{\text{eff}}\mu_0) \rangle_{\text{stoch}}$, which yields $S_{\text{Shannon},\text{eff}} = \{1.375, 0.983, 0.812\}$ for $\zeta = \{0.1, 1, 10\}$ and $\mu_0 = \phi_0$, respectively. Of course, $S_{\text{Shannon},\text{eff}} > 0$ for arbitrary coupling constant provided μ_0 is chosen large enough ($\mu_0 = \phi_0$ suffices). This is so simply because $V_{\text{eff}} > 0$ ($\forall \phi_b, \zeta > 0$). Again, we have seen that the Shannon entropy of the effective field theory gives more reasonable results than the corresponding quantity calculated by the Starobinsky-Yokoyama procedure.

Finally, we give one last cursory look at the V_{eff} shown in figure 3, and observe that in the weak coupling limit (see the $\zeta = 10$ curves on the left panel of figure 3) the effective potential V_{eff} become flatter around the origin ($\phi_b < \phi_0$) as ζ becomes larger, and it becomes steeper for large values of the field ($\phi_b > \phi_0$), such that there is a sudden change in the effective potential curvature at $\phi_b \sim \phi_0$. These features are reminiscent of a Maxwell construction (see *e.g.* Ref. [49]), which is an approximate procedure for constructing the free energy/effective potential close to a critical point (where fluctuations become massless and long range correlations develop). Indeed, in the weak coupling regime one can approximate the effective potential by the following Maxwell-like construction: use Eq. (39) for $\phi_b < \phi_0$ and Eqs. (43–44) for $\phi_b > \phi_0$, and this will give a reasonable approximation to the true effective potential, that exhibits a discontinuity in curvature at $\phi_b \simeq \phi_0$.

C. The $O(N)$ model

In analogy to the convexity condition in the $O(1)$ model (27), in this more general case the effective potential $V_{\text{eff}}(\vec{\phi}^b)$ ($\vec{\phi}^b = (\phi_i^b, i = 1, \dots, N)$) will be convex provided the corresponding Hessian matrix

$$\frac{\partial^2 V_{\text{eff}}(\vec{\phi}^b)}{\partial \phi_i^b \partial \phi_j^b} = \frac{\partial J_i(\vec{\phi}^b)}{\partial \phi_j^b} \quad (48)$$

is strictly positive (*i.e.* all of its eigenvalues are strictly positive), where $\vec{J} = (J_i)$ is the source current ($\rho_{\vec{J}}(\vec{\phi}) = \rho(\vec{\phi})e^{\vec{J} \cdot \vec{\phi}}$). Taking the inverse of the matrix (48) (which exists provided its determinant does not vanish) we get (*cf.* Eqs. (28–32))

$$\left(\frac{\partial^2 V_{\text{eff}}(\vec{\phi}^b)}{\partial \phi_i^b \partial \phi_j^b} \right)^{-1} = \langle \phi_i \phi_j \rangle_{\vec{J}} - \langle \phi_i \rangle_{\vec{J}} \langle \phi_j \rangle_{\vec{J}}. \quad (49)$$

The simplest way of finding the eigenvalues of the inverse Hessian matrix (49) is to make use of the $O(N)$ symmetry, which implies that there exists an orthogonal transformation R ($R \cdot R^T = I = R^T \cdot R$) which

brings the source current \vec{J} to the first component $\vec{J}' = R \cdot \vec{J}$, where $\vec{J}' = (J, 0, \dots, 0)$ and $J = \|\vec{J}\|$. Now, making use of the invariance $\vec{J}^T \cdot \vec{\phi} = (\vec{J}')^T \cdot \vec{\phi}'$, $\vec{\phi}' = R \cdot \vec{\phi}$, and of the fact that the Jacobian of the transformation $\vec{\phi} \rightarrow \vec{\phi}'$ equals unity, we conclude that the partition function $Z(\vec{J}) = Z(\vec{J}')$ (this just means that $Z(\vec{J})$ is a function of \vec{J}^2 , which is invariant under orthogonal transformations). Consequently, we can act with R^T from the left and with R from the right on (49) to obtain a rotated Hessian matrix, $\langle \phi'_i \phi'_j \rangle_{\vec{J}'} - \langle \phi'_i \rangle_{\vec{J}'} \langle \phi'_j \rangle_{\vec{J}'}$, which is in its diagonal form! In order to see that, we shall now write the positivity condition for the nonvanishing elements of the rotated inverse Hessian matrix,

$$\begin{cases} i = j = 1 : \langle (\phi'_1)^2 \rangle_{\vec{J}'} - \langle \phi'_1 \rangle_{\vec{J}'}^2 > 0, \\ i = j \neq 1 : \langle (\phi'_i)^2 \rangle_{\vec{J}'} > 0, \end{cases} \quad (50)$$

where in the last inequality we used the fact that $\langle \phi'_i \rangle_{\vec{J}'} = 0$ for $j = 2, 3, \dots, N$, which is a consequence of the fact that $J'_i = 0$ ($i = 2, 3, \dots, N$). Now the first inequality is proved identically as in the $O(1)$ case, while the second line inequality is trivially true. This completes the proof that V_{eff} is convex, implying that deep infrared scalar fields $\vec{\phi}^b$ on de Sitter cannot undergo phase transitions.

Needless to say, one can perform numerical and analytical analysis of the resulting $V_{\text{eff}}(\vec{\phi}^b)$ by using analogous methods as in sections III A and III B.

IV. DISCUSSION

We use the formalism of Starobinsky's stochastic inflation to show in section II that there can be no scalar field condensate formation in an $O(N)$ symmetric scalar field theory. For simplicity we have separately considered the $O(1)$ model (a real scalar field) and the $O(2)$ model (a complex scalar field). Nevertheless, as figure 2 clearly indicates, the probability distribution function for stochastic field that varies over the Hubble scale shows a double-peak structure, implying that first order transitions can exist during inflation, such that one can foresee formation of topological defects on sub-Hubble scales that persist on super-Hubble scales, at least for a while.

In section III we borrow the concept of effective field theories to study the probability distribution of the deep infrared fields in de Sitter space, that 'live' on the asymptotic future timelike infinity i^+ of de Sitter. We prove that the effective potential that governs the probability distribution of deep infrared fields of an $O(N)$ scalar field theory must be *strictly convex*. Our proof of convexity holds for arbitrary integer N and for arbitrary values of the background field. Our proof applies for an arbitrary quartic (Hubble scale renormalized) potential, and we strongly suspect that, by making use of analogous methods, one can construct a proof of convexity for an arbitrary $O(N)$ symmetric local scalar field theory. An important

consequence of this result is the impossibility of field condensation at the asymptotic future timelike infinity i^+ of de Sitter space illustrated in figure 1. The physical reason for this symmetry restoration is the strong infrared vacuum fluctuations on de Sitter space.

It is worth mentioning that the effective potential can be fully characterized in terms of just one physical dimensionless parameter ζ defined in (20), which characterizes the inverse coupling strength. Furthermore, while for small background field values the effective potential is Gaussian (corresponding to a positive mass parameter [5, 13]) for large field values it reduces, as expected, to the tree level quartic potential plus a correction that logarithmically depends on the background field. On the other hand, as our analysis in section IIIB shows, the curvature of V_{eff} is much softer than the one implied by the Starobinsky-Yokoyama approach.

In section IIIB we also discuss the backreaction on the background geometry, and show that it is always positive and suppressed independently on the choice of parameters. The backreaction is typically of the order $\Delta\Lambda/\Lambda = \mathcal{O}(\Lambda/m_{\text{p}}^2)$, which is to be contrasted with the result obtained by the Starobinsky-Yokoyama procedure, which can be either positive or negative (in the limit when $\zeta \rightarrow \infty$, $\langle V \rangle_{\text{stoch}} \rightarrow -2\zeta$).

Since astronomical observations of cosmic microwave background anisotropies and Universe's large scale structure measure some fixed finite physical scales, it would be of interest to extend the analysis in this paper to study a real scalar field probability distribution, and the corresponding field correlators, on some finite physical scale $\mu \ll H$. In order to properly study these, one would have to develop a full effective theory for the probability distribution $\rho_{\text{eff}}(\phi, \mu, t)$ on some finite scale μ , for which a first – but possibly naïve [55, 56] – guess is given by Eq. (2). Since $\rho_{\text{eff}}(\phi, \mu, t)$ is the classical equivalent of the density operator of quantum field theory, $\rho_{\text{eff}}(\phi, \mu, t)$ must contain complete information about the theory, that includes (equal time) field correlators, which are of relevance for cosmology.

Finally, it would be of interest to extend our analysis to other stochastic theories whose evolution is of the Langevin type. Interesting examples include the infrared dynamics of thermal field theories such as non-Abelian gauge theories [57, 58].

Appendix

O(N) case

Here we show that the function $F(\zeta, N)$ in Eq. (23) is strictly negative for any positive integer N and real $\zeta > 0$. Let us begin by recalling the series around $z = 0$ of the confluent hypergeometric function

(which has infinite radius of convergence):

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)} \frac{z^n}{n!}, \quad (51)$$

where $\Gamma(z)$ denotes the gamma function, $\Gamma(z+1) = z\Gamma(z)$. It is now convenient to rewrite $NF(\zeta, N) = A(\zeta, N)/B(\zeta, N)$, where:

$$B = \Gamma\left(\frac{N}{4}\right) \times {}_1F_1\left(\frac{N}{4}; \frac{1}{2}; \zeta\right) + 2\sqrt{\zeta} \Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{3}{2}; \zeta\right) \quad (52)$$

and

$$\begin{aligned} A = & N\Gamma\left(\frac{N}{4}\right) \times {}_1F_1\left(\frac{N}{4}; \frac{1}{2}; \zeta\right) + 2N\sqrt{\zeta}\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{3}{2}; \zeta\right) \\ & - 2(N+2)\Gamma\left(\frac{N}{4}+1\right) \times {}_1F_1\left(\frac{N}{4}+1; \frac{3}{2}; \zeta\right) - \frac{N+2}{\sqrt{\zeta}}\Gamma\left(\frac{N+2}{4}\right) \times {}_1F_1\left(\frac{N+2}{4}; \frac{1}{2}; \zeta\right). \end{aligned} \quad (53)$$

Let us first consider $B(\zeta, N)$ in Eq. (52). In order to show that $B > 0$ for any $\zeta > 0$ and for any positive integer N , observe first that the confluent hypergeometric function ${}_1F_1(a; b; z) > 0$ if $a > 0$, $b > 0$ and $z > 0$. This is so simply because all of the coefficients in the series (51) contain gamma functions of positive arguments, which are strictly positive. Now, because Eq. (52) contains confluent hypergeometric functions with positive indices a, b and of a positive argument $\zeta > 0$, it follows that $B(\zeta, N) > 0$ for any $\zeta > 0$ and any positive integer N .

To complete the proof we need to show that the function A in (53) is strictly negative. In order to show that, observe that one can combine the first with the third and the second with the fourth term, to obtain,

$$\begin{aligned} A = & -\frac{4}{\sqrt{\zeta}}\Gamma\left(\frac{N}{4} + \frac{3}{2}\right) - \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{N}{4} + n\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} + n\right)} \frac{\zeta^n}{n!} \left[\frac{N^2}{2} + 4n\right] \\ & - \sqrt{\zeta} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{N}{4} + \frac{1}{2} + n\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} + n\right)} \frac{\zeta^n}{(n+1)!} \left[\frac{N^2}{2} + 4n + 2\right] < 0, \end{aligned} \quad (54)$$

which is strictly negative since all three terms are negative for arbitrary $\zeta > 0$ and positive integer N , completing the proof. Notice that the first term in (54) originates from the first term in the series of the last hypergeometric function in (53), and that all gamma functions appearing in (54) are strictly positive.

Asymptotic effective potential

In this appendix we present a derivation of the effective potential (43) for large field values $\phi_b \gg \phi_0$. In doing so we shall make use of the formulae (24–26) and (29–30) in the main text. Since the mapping $J \mapsto \phi_b$ is a monotonically increasing function, when ϕ_b is large so is J . When summing the series (29),

observe that in the limit when J is large, terms with a large n dominate the sum, and hence we need a large index expansion of the confluent hypergeometric function, which can be, for example, found in Eqs. (9.228) and (9.220.3) of Ref. [59],

$${}_1F_1(\alpha; \beta; z) \underset{|\alpha| \rightarrow \infty}{\sim} \frac{\Gamma(\beta) e^{z/2} z^{\frac{1-2\beta}{4}}}{\sqrt{\pi}} \times \cos \left(2\sqrt{\left(\frac{\beta}{2} - \alpha\right)} z + \left(\frac{1}{4} - \frac{\beta}{2}\right) \pi \right), \quad (55)$$

from which we infer,

$${}_1F_1\left(\frac{n}{2} + \frac{1}{4}; \frac{1}{2}; \zeta\right) \underset{n \rightarrow \infty}{\sim} e^{\zeta/2} \cosh(\sqrt{2n\zeta}); \quad {}_1F_1\left(\frac{n}{2} + \frac{3}{4}; \frac{3}{2}; \zeta\right) \underset{n \rightarrow \infty}{\sim} e^{\zeta/2} \frac{\sinh(\sqrt{2n\zeta})}{\sqrt{2n\zeta}}. \quad (56)$$

Inserting this into (29–30) yields

$$Z(j) \equiv e^{-W(j)} = \sum_{n=0}^{\infty} e^{-\omega(n, \zeta, j)} \quad (57)$$

$$\omega(n, \zeta, j) = -\sqrt{2n\zeta} - \frac{n}{2} \ln \left(\frac{j^4}{\zeta} \right) - \ln \left[\frac{\Gamma(\frac{n}{2} + \frac{1}{4}) + \sqrt{\frac{2}{n}} \Gamma(\frac{n}{2} + \frac{3}{4})}{\Gamma(2n+1)} \right] + \ln \left[2\pi\zeta^{1/4} \left(I_{\frac{1}{4}}\left(\frac{\zeta}{2}\right) + I_{-\frac{1}{4}}\left(\frac{\zeta}{2}\right) \right) \right],$$

where we have used a rescaled (dimensionless) current $j = \phi_0 J$ (similarly, we shall be using a dimensionless background field $\varphi_b = \phi_b / \phi_0$). The sum (57) is hard to evaluate. However, by noticing that in the limit of large j , the sum is dominated by large n 's, we can replace the sum by an integral, $\sum_{n=0}^{\infty} \rightarrow \int_0^{\infty} dn$, and make a saddle point approximation to the integral. This means that we can use a large n expansion of $\omega(n, \zeta, j)$, and approximate it by its expansion around the stationary point n_0 , where $\partial\omega/\partial n = 0$,

$$\omega(n, \zeta, j) \approx \omega_0(\zeta, j) + \frac{1}{2} \omega_0''(\zeta, j) (n - n_0)^2, \quad (58)$$

where (in the limit when $n \rightarrow \infty$),

$$n_0 \approx \frac{1}{2} \left(\frac{j^4}{4\zeta} \right)^{1/3} \left[1 + \frac{1}{3} \left(\frac{4\zeta}{j} \right)^{2/3} - \left(\frac{4\zeta}{j^4} \right)^{1/3} \right] \quad (59)$$

The resulting Gaussian integral is convergent if $\omega_0''(\zeta, J) > 0$, which is indeed the case, and it evaluates to,

$$Z(j) = e^{-W(j)} \approx e^{-\omega_0} \sqrt{\frac{2\pi}{\omega_0''}} \left[1 - \frac{1}{2} \operatorname{erfc} \left(\sqrt{\omega_0''/2} \times n_0 \right) \right] \quad (60)$$

where the complement error function $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ can be neglected when the argument is large because, in the large argument limit, $\operatorname{erfc}(x) \sim e^{-x^2}/[\sqrt{\pi}x]$. The standard expression now yields the following approximate expression for φ_b ,

$$\varphi_b = \frac{\partial \ln[Z(j)]}{\partial j} \approx \left(\frac{j}{4\zeta} \right)^{1/3} \left[1 + \frac{1}{3} \left(\frac{4\zeta}{j} \right)^{2/3} - \frac{1}{3} \left(\frac{4\zeta}{j^4} \right)^{1/3} \right], \quad (61)$$

which when inverted gives,

$$j \approx 4\zeta\varphi_b^3 \left[1 - \frac{1}{\varphi_b^2} + \frac{1}{4\zeta\varphi_b^4} \right]. \quad (62)$$

Inserting this into Eq. (25) and making use of (60) yields the asymptotic effective potential

$$\frac{8\pi^2}{3H^4} V_{\text{eff}}(\phi_b) \approx \zeta \left(\varphi_b^4 - 2\varphi_b^2 + \frac{1}{2} \right) + \frac{1}{2} \ln \left(\frac{3\pi\zeta\varphi_b^2}{2} \right) + \ln \left(I_{\frac{1}{4}} \left(\frac{\zeta}{2} \right) + I_{-\frac{1}{4}} \left(\frac{\zeta}{2} \right) \right), \quad (\varphi_b \gg 1), \quad (63)$$

which is used in section IIIB, see Eq. (43).

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- [61] For any two functions f and g of stochastic variable ϕ distributed according to a *positive* probability distribution function $\rho = \rho(\phi)$, if the integral $\langle fg \rangle = \int f(\phi)g(\phi)\rho(\phi)d\phi$ exists, then $\langle fg \rangle$ is a well defined (commutative, bilinear and positive-definite) scalar product. Then the *Cauchy-Schwarz theorem* holds, according to which the following inequality must be satisfied:

$$\langle fg \rangle^2 \leq \langle f^2 \rangle \langle g^2 \rangle .$$

Equality holds only when f and g are linearly dependent, *i.e.* when $f \propto g$.